Lecture 13. Higher-order Linear Equations

In this section, we will generalize the results we discussed in the previous lectures. Here is an outline:

1. General Solutions of Linear Equations

1.1 Linearly Independent Solutions

Definition of linearly dependent/independent

The n functions $f_1, f_2, \cdots, f_n$ are said to be linearly dependent on the interval I if there exist constants $c_1, c_2, \cdots, c_n$ not all zero such that

$$c_1 f_1 + c_2 f_2 + \cdots + c_n f_n = 0$$

for all x in I.

The n functions $f_1, f_2, \cdots, f_n$ are said to be linearly independent on the interval I if they are not linearly dependent. Equivalently, they are linearly independent on I if

$$c_1 f_1 + c_2 f_2 + \cdots + c_n f_n = 0$$

holds on I only when

$$c_1 = c_2 = \cdots = c_n = 0.$$

Example 1 Show **directly** that the given functions are linearly dependent on the real line.

(1) $f(x) = 3, \quad g(x) = 2\cos^2 x, \quad h(x) = \cos 2x$

(2) $f(x) = 5, \quad g(x) = 2 - 3x^2, \quad h(x) = 10 + 15x^2$

ANS: (1) By def, we need to find C_1, C_2, C_3 not all zeros such that

$$C_1 f(x) + C_2 g(x) + C_3 h(x) = 0$$

$$\Rightarrow C_1 \cdot 3 + C_2 \cdot \underbrace{2\cos^2 x}_{= \cos 2x + 1} + C_3 \cos 2x = 0$$

$$\Rightarrow C_1 \cdot 3 + C_2 (\cos 2x + 1) + C_3 \cos 2x = 0$$

Let $C_2 = 1$, $C_3 = -1$ we have

$$C_1 \cdot 3 + \cancel{\cos 2x} + 1 - \cancel{\cos 2x} = 0 \Rightarrow C_1 = -\frac{1}{3}$$

Thus $-\frac{1}{3} \cdot 3 + 1 \cdot 2\cos^2 x - \cos 2x = 0$

So $f(x), g(x), h(x)$ are linearly dependent by def.

(2). We need to find C_1, C_2, C_3 not all zeros such that

$$C_1 f + C_2 g + C_3 h = 0$$

$$\Rightarrow C_1 \cdot 5 + C_2 (2 - 3x^2) + C_3 (10 + 15x^2) = 0$$

Let $C_2 = 5$, $C_3 = 1$, then

$$C_1 \cdot 5 + 10 - 15x^2 + 10 + 15x^2 = 0 \Rightarrow C_1 5 = -20$$

$\Rightarrow C_1 = -4$

Thus $-4 \cdot 5 + 5(2 - 3x^2) + 1 \cdot (10 + 15x^2) = 0$

Wronskian of n functions

Suppose that the n functions $f_1, f_2, \cdots, f_n$ are all $n-1$ times differentiable. Then their **Wronskian** is the $n \times n$ determinant

$$W(x) = W(f_1, f_2, \cdots, f_n) = \begin{vmatrix} f_1 & f_2 & \cdots & f_n \\ f_1' & f_2' & \cdots & f_n' \\ \vdots & \vdots & & \vdots \\ f_1^{(n-1)} & f_2^{(n-1)} & \cdots & f_n^{(n-1)} \end{vmatrix}.$$

- The Wronskian of n **linearly dependent** functions $f_1, f_2, \cdots, f_n$ is **identically zero**.

  Idea of the proof:

  - We show for the case $n = 2$. The case for general n is similar.
  - If f_1 and f_2 are linearly dependent, then $c_1 f_1 + c_2 f_2 = 0 \quad (*)$ has nontrivial solutions for c_1 and c_2 (c_1 and c_2 are not all zeros).
  - We also have $c_1 f_1' + c_2 f_2' = 0$ from $(*)$.
  - Thus we have the linear system of equations

    $$\begin{aligned} c_1 f_1 + c_2 f_2 &= 0 \\ c_1 f_1' + c_2 f_2' &= 0 \end{aligned}$$

  - By a theorem in linear algebra, the above system of equations has nontrivial solutions for $\begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$ if and only if the determinant of the coefficient matrix is 0, that is,

    $$\begin{vmatrix} f_1 & f_2 \\ f_1' & f_2' \end{vmatrix} = 0$$

- So to show that the functions $f_1, f_2, \cdots, f_n$ are **linearly independent** on the interval I, it suffices to show that their Wronskian is **nonzero at just one point of I**.

1.2. n-th order linear differential equation

The general ***n*th-order linear** differential equation is of the form

$$P_0(x)y^{(n)} + P_1(x)y^{(n-1)} + \cdots + P_{n-1}(x)y' + P_n(x)y = F(x).$$

We assume that the coefficient functions $P_i(x)$ and $F(x)$ are continuous on some open interval I.

Homogeneous linear equation

Similar to Lecture ~~8~~ 9, we consider the **homogeneous linear equation**

$$y^{(n)} + p_1(x)y^{(n-1)} + \cdots + p_{n-1}(x)y' + p_n(x)y = 0 \tag{2}$$

Theorem 1 Principle of Superposition for Homogeneous Equations

Let $y_1, y_2, \cdots y_n$ be n solutions of the homogeneous linear equation (1) on the interval I. If $c_1, c_2, \cdots, c_n$ are constants, then the linear combination

$$y = c_1y_1 + c_2y_2 + \cdots + c_ny_n$$

is also a solution of Eq. (1) on I.

Theorem 4 General Solutions of Homogeneous Equations

Let $y_1, y_2, \cdots y_n$ be n linearly independent solutions of the homogeneous equation

$$y^{(n)} + p_1(x)y^{(n-1)} + \cdots + p_{n-1}(x)y' + p_n(x)y = 0 \tag{1}$$

on an open interval I where the p_i are continuous. If Y is any solution of Eq. (1), then there exist numbers $c_1, c_2, \cdots, c_n$ such that

$$Y(x) = c_1y_1(x) + c_2y_2(x) + \cdots + c_ny_n(x)$$

for all x in I.

Example 2 Use the Wronskian to prove that the given functions are linearly independent on the indicated interval.

$f(x) = e^x, \quad g(x) = \cos x, \quad h(x) = \sin x; \quad$ the real line

Remark: 3×3 matrix determiant:

$$\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = a \begin{vmatrix} e & f \\ h & i \end{vmatrix} - b \begin{vmatrix} d & f \\ g & i \end{vmatrix} + c \begin{vmatrix} d & e \\ g & h \end{vmatrix}$$

$$= a(ei - fh) - b(di - fg) + c(dh - eg)$$

ANS: By the previous page, we know it suffices to show that $W(f, g, h) \neq 0$ at just one point on the real linea.

$$W(f,g,h) = \begin{vmatrix} f & g & h \\ f' & g' & h' \\ f'' & g'' & h'' \end{vmatrix} = \begin{vmatrix} e^x & \cos x & \sin x \\ e^x & -\sin x & \cos x \\ e^x & -\cos x & -\sin x \end{vmatrix}$$

$$= e^x \begin{vmatrix} -\sin x & \cos x \\ -\cos x & -\sin x \end{vmatrix} - \cos x \begin{vmatrix} e^x & \cos x \\ e^x & -\sin x \end{vmatrix} + \sin x \begin{vmatrix} e^x & -\sin x \\ e^x & -\cos x \end{vmatrix}$$

$$= e^x (\overbrace{\cancel{\sin^2 x + \cos^2 x}}^{1}) - \cos x(-e^x \sin x - e^x \cos x) + \sin x(-e^x \cos x + e^x \sin x)$$

$$= e^x + \cancel{e^x \cos x \sin x} + \underline{e^x \cos^2 x} - \cancel{e^x \sin x \cos x} + \underline{e^x \sin^2 x}$$

$$= e^x + e^x (\overbrace{\cancel{\sin^2 x + \cos^2 x}}^{1}) = 2e^x \neq 0 \quad \text{for all } x \in \mathbb{R}.$$

Thus $f(x)$, $g(x)$, $h(x)$ are linearly independent.

Higher-order Homogeneous Equations with Constant Coefficients

Recall in Lecture 10 and Lecture 11, we talked about 2nd-order homogeneous equations with constant coefficients of the following form

$$ay'' + by' + cy = 0$$

To solve for y, we first solve for r from the **characteristic equation**

$$ar^2 + br + c = 0,$$

which has roots $r_1, r_2 = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$.

Case 1. r_1, r_2 are real and $r_1 \neq r_2$ ($b^2 - 4ac > 0$):

$$\text{General solution: } y = c_1 e^{r_1 x} + c_2 e^{r_2 x}$$

Case 2. r_1, r_2 are real and $r_1 = r_2$ ($b^2 - 4ac = 0$):

$$\text{General solution: } y = (c_1 + c_2 x)e^{r_1 x}$$

Case 3. r_1, r_2 are complex numbers ($b^2 - 4ac < 0$):

We can write $r_{1,2} = A \pm Bi$.

$$\text{General solution: } y = e^{Ax}\left(c_1 \cos Bx + c_2 \sin Bx\right)$$

In this lecture, we will discuss how to solve the general homogeneous equations with constant coefficients of the form

$$a_n y^{(n)} + a_{n-1} y^{(n-1)} + \cdots + a_2 y'' + a_1 y' + a_0 y = 0 \tag{1}$$

Similar to 2nd-order homogeneous equations, we look at the corresponding **characteristic equation:**

$$a_n r^n + a_{n-1} r^{n-1} + \cdots + a_2 r^2 + a_1 r + a_0 = 0 \tag{2}$$

We have 3 cases of the roots for Eq (2).

1. Distinct real roots
2. Repected real roots
3. Complex roots
    - distinct
    - repeated

The results in Lecture 9 and 10 can be generalized in a natural way:

Case 1. Distinct Real Roots

If the roots $r_1, r_2, \cdots, r_n$ of $Eq(2)$ are real and distinct, then

$$y(x) = c_1 e^{r_1 x} + c_2 e^{r_2 x} + \cdots + c_n e^{r_n x}$$

Case 2. Repeated Real Roots

If Eq (2) has repeated root r with multiplicty k, then the part of a general solution of $Eq(1)$ corresponds to r is

$$\left(c_1 + c_2 x + c_3 x^2 + \cdots + c_k x^{k-1}\right) e^{rx}$$

Case 3. Complex Roots

Unrepeated complex roots: If $r_{1,2} = A \pm Bi$ are roots of the characteristic equation, then the corresponding part to the general solution

$$y = e^{Ax}(c_1 \cos Bx + c_2 \sin Bx)$$

Repeated complex roots

If the conjugate pair $a \pm bi$ has multiplicity k, then the corresponding part of the general solution has the form

$$\begin{aligned}&\left(A_1 + A_2 x + \cdots + A_k x^{k-1}\right) e^{(a+bi)x} + \left(B_1 + B_2 x + \cdots + B_k x^{k-1}\right) e^{(a-bi)x} \\ &= \sum_{p=0}^{k-1} x^p e^{ax} \left(c_p \cos bx + d_p \sin bx\right)\end{aligned}$$

Example 3 Find the general solution to the given differential equation.

$$y^{(3)} - 7y'' + 12y' = 0$$

ANS: The corresponding char. eqn. is

$$r^3 - 7r^2 + 12r = 0$$

$$\Rightarrow r(r^2 - 7r + 12) = 0$$

$$\Rightarrow r(r-3)(r-4) = 0$$

$$\Rightarrow r_1 = 0, \ r_2 = 3, \ r_3 = 4$$

Thus the general solution is $y(x) = c_1 \cancel{e^{0x}}^{\,1} + c_2 e^{3x} + c_3 e^{4x}$

$= c_1 + c_2 e^{3x} + c_3 e^{4x}$

Example 4.

A 9th order, linear, homogeneous, constant coefficient differential equation has a characteristic equation which factors as follows.

9 linearly independent sols $\quad (r^2+4r+8)^2 r^2 (r-1)^3 = 0 \quad \circledast$

Write the nine fundamental solutions to the differential equation as functions of the variable t.

ANS: We consider each factor appears in Eq $\circledast$.

- $r^2 = 0$ implies the corresponding solutions are

$$(c_1 + c_2 t)e^{0t} = c_1 + c_2 t$$

This gives the fundamental solution

$$y_1(t) = 1, \quad y_2(t) = t$$

- $(r-1)^3 = 0$ gives 3 repeated roots $r=1$. Then the corresponding solutions are

$$(C_3 + C_4 t + C_5 t^2)e^t$$

This gives $y_3(t) = e^t$, $y_4(t) = te^t$, $y_5(t) = t^2 e^t$

$\sqrt{-16} = 4i$

- $(r^2+4r+8)^2 = 0 \Rightarrow r = \dfrac{-4 \pm \sqrt{16-32}}{2} = -2 \pm 2i$

each repeated twice. The corresponding solution is

$$(C_6 + C_7 t)e^{-2t}\cos 2t + (C_8 + C_9 t)e^{-2t}\sin 2t$$

Thus $y_6(t) = e^{-2t}\cos 2t$, $y_7(t) = te^{-2t}\cos 2t$,

$y_8(t) = e^{-2t}\sin 2t$, $y_9(t) = t\,e^{-2t}\cos 2t$.

Example 5 Find a general solution the differential equation.

$$y^{(4)} + 3y^{(3)} + 3y'' + y' = 0$$

ANS: The corresponding char. eqn. is

$$r^4 + 3r^3 + 3r^2 + r = 0$$

$$\Rightarrow r(r^3 + 3r^2 + 3r + 1) = 0 \quad \circledast$$

Notice $r = -1$ is a solution.

$$\circledast \Rightarrow r(r-(-1))(\underline{r^2 + 2r + 1}) = 0$$

Long Division of Polynomials.

$$\begin{array}{r|l} & r^2 + 2r + 1 \\ \hline r+1 & r^3 + 3r^2 + 3r + 1 \\ & r^3 + r^2 \\ \hline & 2r^2 + 3r + 1 \\ & 2r^2 + 2r \\ \hline & r + 1 \\ & r + 1 \\ \hline & 0 \end{array}$$

$$\circledast \Rightarrow r(r+1)(r^2 + 2r + 1) = 0$$

$$\Rightarrow r(r+1)^3 = 0$$

$$\Rightarrow r_1 = 0, \quad r_2 = r_3 = r_4 = -1$$

Thus the general solution is

$$y(x) = C_1 e^{r_1 x} + (C_2 + C_3 x + C_4 x^2) e^{r_2 x}$$

$$\Rightarrow y(x) = C_1 \cancel{e^{0 \cdot x}}^{\,1} + (C_2 + C_3 x + C_4 x^2) e^{-x}$$

Exercise 6. Suppose that a fourth order differential equation has a solution $y = 9e^{4x} x \cos(x)$.

(a) Find such a differential equation, assuming it is homogeneous and has constant coefficients.

(b) Find the general solution to this differential equation.

Solution.

(a) We know a solution is of the form $y = 9e^{4x} x \cos(x)$, so we can figure out the corresponding root r to the characteristic equation.

Note the scalar 9 does not matter as the equation is homogeneous.

The x in $e^{4x} x \cos(x)$ means the solution to the characteristic equation has repeated roots and repeated twice.

The part $e^{4x} \cos(x)$ indicates there is a pair of complex root of type $A \pm iB$, where $A = 4$ and $B = 1$. Thus we have $r_{1,2} = 4 \pm i$ with multiplicity 2.

Therefore the charactersitic equation is

$$(r - (4+i))^2 (r - (4-i))^2 = r^4 - 16r^3 + 98r^2 - 272r + 289 = 0$$

Thus the answer is $r^4 - 16r^3 + 98r^2 - 272r + 289 = 0$.

(b) By the Equation (3) after case 3, we have the general solution as

$$y = (c_1 + c_2 x)e^{4x} \sin(x) + (c_3 + c_4 x)e^{4x} \cos(x)$$